Sparse representation of intricate natural image with multi-scale geometric dictionary

Xingyu Yang[1], Jinshan Su[1*], Jing Ma[1], Zhengfang Deng[1], Jing Jin[2]

[1]*College of Electronic and Information Engineering, Yili Normal University, No.448, Jiefang West Road, Yining, China*

[2]*School of Electronic Science and Engineering, Nanjing University, No.163, Xianlin Dadao, Nanjing, China*

Received 24 November 2014, www.cmnt.lv

Abstract

Sparse representation of natural image is the fundamental problem of multi-scale geometric analysis, deep learning and K-SVD learning method. Traditional multi-scale geometric analysis is based on simple mathematical model which cannot express intricate natural images, and learning methods rely on prior knowledge. In this paper, a complex sparse representation mathematical model of natural images which have non-smooth area, non-smooth contours and intricate texture features is proposed. The model is established from the perspective of highly nonlinear approximation and according to the theories of wavelet, ridgelet, contourlet, and dictionaries such as wavelet dictionary and multi-scale ridgelet dictionary. The model can represent all natural images without any learning and priori knowledge. Simulation comparison experiments which established by a new multi-scale geometric dictionary show that this model greatly improves the sparse ratio and peak signal noise ratio and has the progressive optimal expression of intricate natural images.

Keywords: intricate natural images, mathematical model, multi-scale geometric dictionary

1 Introduction

Currently, wavelet dictionary is one of common used method of image sparse representation. Wavelet can provide the sparsest representation of the signal which has point singularity [1]. However, for the singularity of two-dimensional image signal is mainly generated by edge and contour. Since the wavelet is the tensor product of two orthogonal wavelets, the number of selecting directions is three, which are the horizontal, vertical, diagonal directions. These directions cannot sparsely represent contour and edge information of image. Therefore, there are some limitations when wavelet processing a two-dimensional image. To overcome the limitations and solve the sparse representation problem of high dimensional singularity complex natural images, multi-scale geometric analysis (MGA) was proposed. In recent years, common multi-scale geometric transformation methods include the ridgelet transform [2], the curvelet transform, the contoulet transform, the bandelet Transform and shearlet transform [3] and so on. Based on the simple mathematical model of multi-scale geometric analysis, a more complicated mathematical model is proposed in this paper. This model is suitable for sparsely representing two-dimensional natural image which has highly nonlinearity and complex textures regions. According to this model, an improved dictionary which is more superior than wavelet dictionary [4] is constructed, which can approximate complex two-dimensional natural images more effective for overcoming the limitations of wavelet dictionary. Moreover, comparing with deep learning methods [5] and K-SVD [6] learning method, the method proposed in this paper can get the optimal representation of intricate natural image without any prior knowledge.

2 Experimental section

2.1 NONLINEAR WAVELET APPROXIMATION OF THE IMAGE

Whether exploring the theory or practical application, wavelet analysis is a powerful tool to study nonlinear approximation, which has the best approximation performance for some function classes (if any sector variogram class). Nonlinear wavelet approximation used in signal processing and image processing is the most common.

Consider the decomposition [7] of the function $f \in L^p(R)$ under the biorthogonal wavelet as follows:

$$f = \sum_{k \in z} \sum_{j \in z} 2^k < f,\ \tilde{\varphi}(2^k \bullet - j) > \varphi(2^k \bullet - j), \tag{1}$$

$\tilde{\varphi}$ [8] is a scaling function dual function, and satisfies:

$$\int_R \phi(x-j)\tilde{\phi}(x-k)dx = \delta_{jk}, \tag{2}$$

φ is wavelet function, which is the corresponding scaling function φ. $\tilde{\varphi}$ is wavelet function, which is the corresponding scaling function $\tilde{\varphi}$. For $j \in Z^d, k \in Z$,

* ***Corresponding author's*** e-mail: sqsjs1968@aliyun.com

R^d is represented by $I = 2^{-k}(j+\Omega)$ binary cube, and $\Omega:=[0,1]^d$ denoted as $\eta_I(x):=|I|^{-1/2}\eta(2^K \bullet - j)$. D shows all the sides with a length of a 2^{-K} collection binary cube, then Equation (1) can be written as:

$$f = \sum_{I \in D} C_I(f)\varphi_I, C_I(f) = < f, \tilde{\varphi}_I >. \tag{3}$$

When $\Lambda \in D$ and $\#\Lambda \le n$, then $s = \sum_{i \in \Lambda} a_I \phi_I$, obviously, the aggregate of functions S constitutes a nonlinear space, denoted the space as $\sum_n^{\omega}$ and then the function f in nonlinear n items wavelet space mectric approximation error [9,10] as follows:

$$\sigma_n^{\omega}(f)_p := \inf \|\text{f - s}\|_{L^p(R)}. \tag{4}$$

As the function $f \in L^p(R), 1 < p < \infty, s > 0$, there are inequality of Jackson [11] and Bernstein [12] as following if the Wavelet coefficients $(C_I)_{I \in D}$ belong to l_τ space.

$$\sigma_n(f)_p \le cn^{-s} |f|_{B_\tau^s(L_\tau(R))}, \tag{5}$$

$$\|f\|_{B_\tau^s(L_\tau(R))} \le cn^s \|f\|_{L^p(R)}. \tag{6}$$

So we can characterize the nonlinear wavelet approximation space:

$$A_q^{\gamma/d}(L^p(R^d)) = (L^p(R^d), B_\tau^s(L_\tau(R^d)))_{\gamma/s,q}. \tag{7}$$

2.2 SIMPLE IMAGE MODEL

Nonlinear approximation of functions in high-dimensional space $L^2[0,1]^d$ can be used separable wavelet basis. Considering the two-dimensional (still images), wavelet is C^q in $L^2[0,1]$ and possesses q vanishing moments when

$\psi_{j,n}^1 = \phi_{j,n}(x_1)\psi_{j,n}(x_2)$, $\psi_{j,n}^2 = \psi_{j,n}(x_1)\phi_{j,n}(x_2)$,

$\psi_{j,n}^3 = \psi_{j,n}(x_1)\psi_{j,n}(x_2)$,

then getting as follows:

$$B = (\{\phi_J^2, \text{n}\}_{2^J n \in [0,1]^2}) \cup (\{\psi_j^l, \text{n}\}_{j \le J,\, 2^j n \in [0,1]^2 \quad 1 \le l \le 3}). \tag{8}$$

It forms a standard set of orthogonal groups in $L^2[0,1]^2$. If the function $f(x_1, x_2) \in L^2[0,1]^2$ is regular consistent, that is $f \in C^\alpha$, the vanishing moments of wave function $p > \alpha$, then $\varepsilon_n^W[\text{M}] = \| f - f_M \|^2 \le CM^{-\alpha}$. Because there is no other group, $\varepsilon_n[\text{M}] = \| f - f_M \|^2 \le CM^{-\beta}$, so this non-linear approximation approach is optimal. In fact, most natural objects are smooth and have smooth edges, thereby establish a simple image model:

$$F_\Gamma(\text{a,A}) = \bigcup_{\Upsilon \in \Gamma(\text{s,C})} F_\Upsilon(\text{a,A}), \tag{9}$$

where:

$$F_\Gamma(\text{a,A}) = \{ f \in [0,1]^2 \setminus \Upsilon[0,1], \| f \|_{C^a} \le A \}. \tag{10}$$

$$\Gamma(\text{s,C}) = \{ \gamma : [0,1] \to [1/10, 9/10]^2, \| \gamma \|_{C^s} \le \text{C} \}. \tag{11}$$

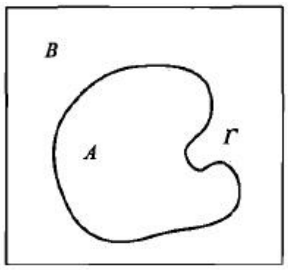

FIGURE 1 Simple image model with smooth edges

Equation (9) represents a class of two-dimensional functions which has a singularity of curves and straight lines. In addition to the curve $\Gamma(\text{s,C})$ is in the two-dimensional plane, it is smooth C^α, and the singular curve $\Gamma(\text{s,C})$ itself is smooth C^s. As shown in Figure 1, the regions A and B are smooth C^a, singular curve is Γ, two order smoothing. Wavelet Transform Based on this simple image model for bounded variation functions and piecewise regular function approximation error can only reach M^{-1} attenuation level is not optimal. Wavelet basis functions from the dictionary are a collection of wavelet transform in accordance with the rules to select a certain linear combination of basic functions to represent or approximate the signal. Wavelet dictionary definition is: In the space of $d_i \in R^{d \times i}$, if the N atoms are from the same wavelet function, $D_f \in R^{d \times N} : d < N$ is composed by a known dictionary atoms wavelet function D_f, these atoms satisfy the maximum separation from the dimensional space, and has a good sparse approximation performance. Assuming that each atom in the dictionary $D_f \in R^{d \times N} : d < N$, the mathematical form is $f(t, \lambda_i)$, where $\Lambda = \{\lambda_i\}_{i=1}^N$ is a set of vectors. These vectors contain a number of functions that represent each wavelet function atoms, whereby an image can be seen that the wavelet is still not optimal Dictionary in order to solve the above problem, we need to create more complex and more comprehensive image model basis functions dictionary.

2.3 IMAGE MODEL COMPLEX FUNCTIONS

Methods currently solving image representation are based on a class of two-dimensional function with curve singularity (including the line). This function apart from the curve in the two-dimensional plane which smooth C^a and singular curve itself is smooth C^s .It is a simple image model consists of bounded variation and amplitude bounded combine. Two-dimensional functions (such as natural images special area) varied, both "point singularity" there "line singularity"; both smooth areas, there are also non-smooth region; both smooth contours, but also non-smooth contours both non-texture area, but also the texture region [13] lines. At present, only a simple mathematical model to deal effectively with piecewise smooth function classes, but for complex function types can't do it.

2.4 SINGULAR COMPLEX NATURAL IMAGES REPRESENTED

Taking into account the effective treatment of high-dimensional (two-dimensional) complex functions *f*, the two dimensional singularity complex natural image has the synthesis expansion Affine Systems (Equation (12)), if basis functions $\phi\ (x)$ for any $f \in L^2(R^2)$ and scale *j, l, k* have the form of

$$\sum_{j,l,k} \left|\left(f, \phi_{j,l,k}\right)\right|^2 = \| f \|^2 .$$

$$\Lambda_{MN}(\phi) = \left\{\phi_{j,l,k}(x) = \left|\det M\right|^{j/2} \phi(N^l M^j x - k)\right\}. \tag{12}$$

Here $\phi \in L^2(R^2)$, the matrix M^j is associated with scale transformation and maintaining constant area associated with geometric transformation, thus constitutes a Parseval frame. The frame can make up the basic elements of various scales, positions and directions as wavelets. The Affine Systems combine with multi-scale geometric analysis can locate the discontinuous curve of function and differentiate their directions. This combination has the characteristics as follow: (1) good localization characteristics; (2) strong direction sensitivity; (3) spatial localization. For one-dimensional signal, the singularity of the image depends on singular point. The continuous wavelet transform $W_f(a,t) = < f, \phi_{\alpha t} >$ of one-dimensional function f can be localized singularity and $\phi_{\alpha t} = a^{-1}\phi(a^{-1}(x-t))$, that is the function $W_f(a,t)$ can quickly tends to zero when a tends to zero and t locates around the singular point or $W_f(a,t)$ slowly tends to zero when t is singular point. For two-dimensional signal, the singularity of the image depends on the singular point and singular lines. We can establish a two-dimensional transform $L_f(a,s,t) = < f, \phi_{ast} >$ as one-dimensional signal, and describe the speed of the declined amplitude of the function $\left|L_f(a,s,t)\right|$ along with the reducing scale parameter *a* as: $\alpha a^r \le \left|L_f(a,s,t)\right| \le \beta a^r, r \in R, 0 < \alpha \le \beta < \infty$ and sign as $L_f(a,s,t) \sim a^r$ .There is a relation $L_f(a,s,t) \sim a^{-3/4}$ for the function $f(x) = \delta(x + x_0)$ with the singular point if *t*=0. And that exists $L_f(a,s,t) \sim a^{-1/4}$ for the function $f(x) = \delta(x + px_2)$ with the singular line if $t_1 = -pt_2$ and $S = P$. We set up a circular area $D = \{(x_1, x_2) \in R^2, x_1^2 + x_2^2 \le 1\}$ and the singular characteristic function of the circular area *D* contains a circular curve. There will be $L_f(a,s,t) \sim a^{-3/4}$ if $t = (t_1, t_2)$ meets $t_1^2 + t_2^2 = 1$ and parameters *S* satisfies $s = t_2 / t_1, t_1 \ne 0$. So that we can locate the position of the singular point and automatically track the singular curve. Then this way can solve the curve singularity problem which traditional wavelet transform can't resolve.

2.5 COMPLEX NATURAL IMAGES REGULAR REPRESENTATION

Based on the representation of the image edge describe the edge geometric regularity which is not critical and difficult to portray a better image. Pennec and Mallat [14] introduce the flow to characterize the geometric properties of the image. They use the direction vector to represent the local image geometry changing direction. These vectors give the local direction in the regular changes of the image. We can improve the performance of image transformation approximation method in the image processing tasks if we are able to know the geometric regularity of the images in advance and to fully utilize them. Pennec and Mall firstly define geometric flow vectors which can characterize the partial regular direction of the image, and then the supported interval *S* of the image are dyadic split $S = U_i \Omega_i$ .Each split interval contains only one contour line (edge) when the mesh is sufficiently thin. The changing of image gray value are consistent regular in the local region Ω_i where do not contain all of the contour line. Therefore, we don't need to define the direction of the geometric vector lines in these areas. We calculate the vector line of the vector field in the area Ω_i according to the local geometry of regular direction and under the constraints of the global optimum. Then the interval which is defined in Ω_i performs wavelets transformation along vector line in order to take full advantage of the local geometry of the image itself regularity. As a result it can constitute an orthonormal basis set in the split region Ω_i and $L^2(\Omega)$:

$$\begin{aligned}&\left\{\varphi_{j,m_1}(x_1 - c(x_2))\varphi_{l,m_2}(x_2), \varphi_{j,m_1}(x_1 - c(x_2))\varphi_{l,m_2}(x_2),\right.\\ &\left.\varphi_{j,m_1}(x_1 - c(x_2))\varphi_{l,m_2}(x_2)\right\}_{j\bullet l > j\bullet m_1, m_2}\ j,l \in Z, k \in Z^i\Big\}\end{aligned} \tag{13}$$

ϕ_{j,m_1} is scaling function and ϕ_{l,m_2} is wavelet function. We can prove that the coefficients generated on the fine-scale is much less than bending wavelet coefficients in this way, if $f(x_1, x_2 + c(x_1))$ is the function C^{α} in Ω for all fixed x_1 and x_2. Through the above comparison, this image processing method has the following characteristics: (1) A good multi-scale nature can ensure continuous refinement image; (2) Frequency localization properties, we can simultaneously achieve precise positioning of time domain and frequency domain; (3) multi-directional and anisotropic make up for the shortcomings of wavelet function; (4) A high order of vanishing moments direction that we can get more sparse representation; (5) A good adaptability, obtaining the required optimal basis functions adaptively. So we can get the conclusion that this method can adaptively track geometric regularity images, obtain more sparse representation and improves the approximation of complex properties of natural images.

The above analysis results can construct a more efficient natural images based on multi-scale highly nonlinear function approximation with complex features sparse representation model:

$$\Lambda_{MN}(\varphi) = \left\{\varphi_{j,l,k}(x) = \left|\det M\right|^{j/2} \varphi(N^l M^j x - k)\right\};$$
$$\left\{\varphi_{j,m_1}(x_1 - c(x_2))\varphi_{l,m_2}(x_2), \varphi_{j,m_1}(x_1 - c(x_2))\varphi_{l,m_2}(x_2), \right. \quad (14)$$
$$\left. \varphi_{j,m_1}(x_1 - c(x_2))\varphi_{l,m_2}(x_2)\}_{j\bullet l > j \bullet m_1, m_2}\; j,l \in Z, k \in Z^i \right\}$$

This model not only can detect all the singular points, satisfy with the image processing method of simple model based on wavelet transform, ridgelet transform and contourlet transform and so on. This model can adaptively track the direction of the singular curve, and can accurately describe the singularity characteristic function with the scale parameter changing. Not only can the model achieve the classic description of multi-scale analysis of high-dimensional signal geometric singularity, but also can track the complex adaptive nature of the image geometric regularity, and overcome the wavelet transform (simple image model) in dealing with the limitations of two-dimensional images showed.

3 Results and discussion

3.1 THE NATURAL REALIZATION OF THE COMPLEX IMAGE REPRESENTATION

Minh. N. Do and Martin Vetterli [15] proposed a two-dimensional representation method of the image - contoulet transform [16] which is a multi-resolution, localized, direction method. Contourlet transform actually spin off multi-scale analysis and direction analysis. Firstly, the multi-scale decomposition of image capture the singular point by LP transform, and then the directional filter (DFB) will be distributed point singular in the same direction into a coefficient. This approach is essentially similar to the segment-based structure to approximate the original image. The transformation is affixed conversion and can make the image processing tasks realize more simple and easier. Contourlet transform also has a wealth basis functions which can contain any integer power of a direction base function, thereby contoulet transform [17, 18] can solve linear singular and singular curves and make them both close to optimal representation. This outline wavelet transform has better image processing effects than the traditional wavelet, multi-scale ridge wave. However, the contourlet approximation does not have the best performance when the edges of the most complex natural images are not C^2 critical. And capturing the coefficients needs larger work result in the effect unsatisfied. Thus we establish the function dictionary by a variety of multi-scale structure of the base based on the model (14) of complex natural images to represent the two-dimensional natural images more comprehensive and optimize a single multi-scale function, wavelet function dictionary and ridgelet function dictionaries and so on.

3.2 ESTABLISH DICTIONARY SIMULATION EXPERIMENT AND RESULTS ANALYSIS

The dictionary in this paper is constructed by using an adjustable manner dictionary [19] chosen method. This method is suitable to construct the known basis functions dictionary and the basis function has a good optional. We have introduced the known basis function of the wavelet, contourlet and strip wavelet in the dictionary. A more complex mixture of more comprehensive dictionary is constructed based on a complex structure improved of the natural image model. The dictionary uses optimal matching search method for signal sparse representation - orthogonal matching algorithm (OMP). The new dictionary and the traditional dictionary constructed by simple model were contrasted the effect of complex natural images by MATLAB image simulation. The paper selected four 512×512×8 standard gray image who are goldhill, boats, cameraman, baboon and the enlarged image of a portion. We take the most meaningful 2.5% coefficient; respectively wavelet function dictionary and the new dictionary approximate the natural images of complex nonlinear. Then we obtain the simulation and experimental results of the sparse image ratio (SR) and peak signal to noise ratio (PSNR). Nonlinear approximation result shows as Figure 2, Figure 3, Figure 4 and the sparse ratio and peak signal to noise ratio as Table 1.

TABLE 1 Sparse ratio and signal to noise of nonlinear approximation

Images	**Wavelet dictionary SR/PSNR**	**New dictionary SR/PSNR**
goldhill	5.45/46.33	7.93/47.62
boats	5.45/44.60	7.93/46.04
cameraman	6.40/46.60	7.23/48.13
baboon	6.40/45.05	7.72/46.70

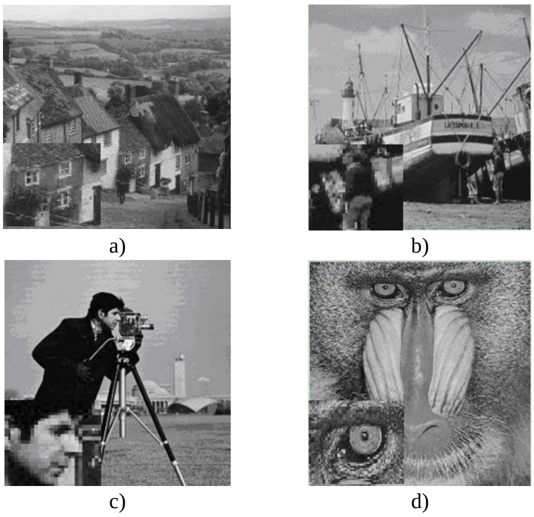

FIGURE 2 Standard image and enlarge portions

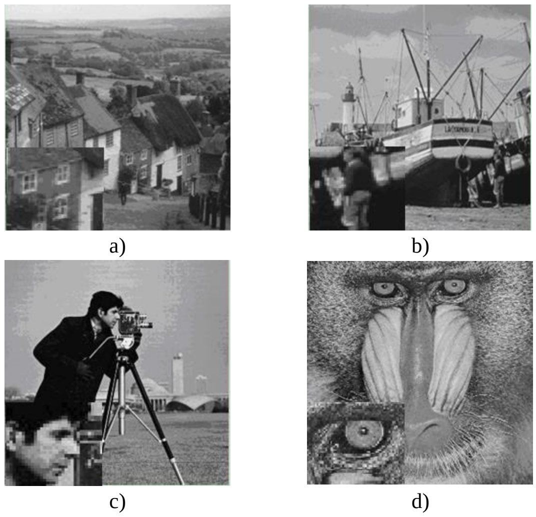

FIGURE 4 Transformed dictionary simulation diagram and enlarge portion

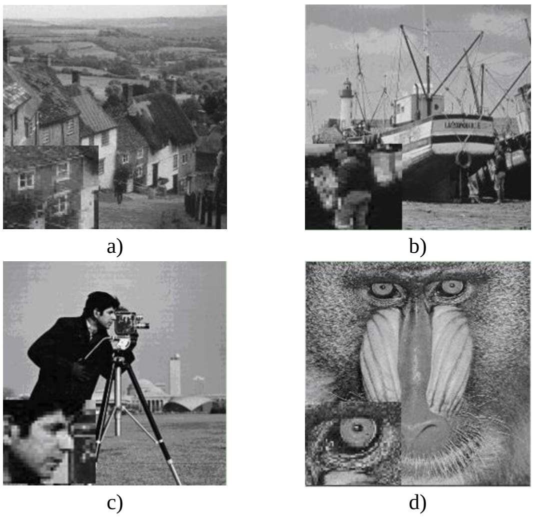

FIGURE 3 The simulation map of wavelet dictionary and enlarged portion

According to depict complex natural images by MATLAB simulation and the comparison of the sparse and peak signal to noise ratio show that the improved dictionary based on the complex natural images model has more advantageous than the traditional dictionary in solving curve singular and image regularity, and approximation results are better in dealing with non-smooth contour, non-smooth area and text region of complex natural images. This can achieve optimal sparse representation progressive.

4 Conclusion

This paper establishes a multi-scale sparse representation model based on highly nonlinear approximation. This model aims at the intricate natural images which have point and line singularity, smooth and non-smooth regions, smooth and non-smooth contours, texture and non-texture areas. According to this model, a new hybrid dictionary is also constructed. Simulation experiments show that this complex natural image model can obtain the optimal sparse representation of the complicated two-dimensional natural images which have non-smooth regions and contour and texture areas. Meanwhile, this method can improve the performance of non-linear approximation of the images and confirmed that the model of complex function image has more superiority than the traditional model of a simple image.

Acknowledgments

This research was financially supported by the Educational Commission of Xingjiang Autonomous Region of China (Grant NO. XJEDU2011I49) and the Fundamental Research Funds for YiLi Normal University (Grant NO. 2011YNZD011).

References

[1] Grossmann A, Morlet J 2009 Decomposition of Hardy functions into square integrable wavelets of constant shape *SIAM Journal on Mathematical Analysis* **15**(1) 722-36

[2] Liang Y T, He L L, Chang H 2010 Effect Evaluation of De-noising for Fundus Image Based on Ridgelet Transform *Opto-Electronic Engineering* **37**(1)

[3] Xu B, Tang Y Y, Fang B 2011 Image texture features classification based on Shearlet transform *Computer Engineering and Applications* **47**(29) 15-7

[4] Rebollo-Neira L, Bowley J, Constantinidesb, Plastinoc A 2012 Self Contained Encrypted Image Folding *Physica A: Statistical Mechanics and its Applications* **391**(23) 5858–70

[5] Zheng Y, Chen Q Q, Zhang Y J 2014 Deep learning and its new progress in object and behavior Recognition *Journal of Image and Graphics* **19**(2) 175-84

[6] Ender M E, Ozden B 2014 K-SVD Meets Transform Learning Transform K-SVD *IEEE Signal Processing Letters* **21**(3)

[7] Xie K, Zhang F 2012 Image denoising method based on dictionary leaning with union of two orthonormal bases *Journal of Computer Application* 2012(4) 1119-21
[8] Dono D L 1993 Uconditional based are optimal based for data compression and for statistical estimation *App. Comput. Harmon. Anal* **1** 100-15
[9] Unser M, Blu T 2003 Mathematical Properties the JPEG2000 wavelet filters *IEEE Trans. on Image Processing* **12** 1080-90
[10] Cohen A, Daubechies I, Guleryuz O G 2002 On the importance of combining wavelet-based nonlinear approximation with coding strategies *IEEE Trans. on Information Theory* **48** 1895-921
[11] Devore R, Jawerth B, Popov V 1992 Compression of wavelet decompositions *Amer. J. Math.* **114** 737-85
[12] Cohen, Devore R, Hochmuth R 2000 Restricted nonlinear approximation 16 85-113
[13] Deng Z Z 2010 Image Denoising Algorithm Based on Multiscale Rigelet Dictionary *Computer Engineering* **36**(23) 207-9
[14] Le Pennec E, Mallat S 2005 Sparase geometric image representation with bandelets *IEEE Trans. on Image Processing* **14**(4) 423-38
[15] Do M N, Vetterli M 2005 The contourlet transform: an efficient directional multiresolution image representation *IEEE Trans. on Image Processing* **14**(12) 2091-116
[16] Stephane M A 2003 Wavelet Tour of Signal Procesing*Yang Lihua, M.Beijing: Machinery Industry Press*
[17] Su J S, Feng Y 2009 An Improved Contoulet Transform and its Application in Image Denoising *Journal of computer-aided design & computer graphics* **21**(11) 1612-7
[18] Cunha E J, Zhou J, Do M N 2006 The nonsubsampled contoulet transform: Theory, design and application *IEEE Trans. on Image Processing* **15**(10) 3089-101
[19] Mairal J, Bach F, Ponce J 2012 Task-driven dictionary learning *IEEE Trans. on Pattern Analysis and Machine Intelligence* **34**(4) 791-804

Authors

Xingyu Yang, China.

Current position, grades: master student of Yili Normal University.
University studies: Harbin Institute of Petroleum (2009-2013).

Jinshan Su, China.

Current position, grades: associate professor, Institute of Electronics and Information Engineering, Yili Normal University.
University studies: Northwestern Polytechnical University (2008-2010), Master on Signal and Information Processing (2010, School of Electronics and Information, Northwestern Polytechnical University).
Scientific interest: signal and information processing.

Jing Ma, China.

Current position, grades: lecturer, Institute of Electronics and Information Engineering, Yili Normal University.
University studies: Northeast Normal university (2006-2009).

Zhengfang Deng, China.

Current position, grades: master student of Yili Normal University.
University studies: Chongqing university of posts and telecommunications (2005-2009).

Jing Jin, China.

Current position, grades: PhD candidate of Nanjing University.
University studies: Nanjing University (2010-2012).
Scientific interest: pattern recognition, image processing.